

Generalizations of Quantum Discord

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The original definition of quantum discord of bipartite states was defined under projective measurements. In this letter we generalize it in two ways: one is we define the quantum discord as the minimal loss of conditional entropy under all one-side general measurements; the other is similar with the original case but we perform the projective measurements on an extended infinite dimensional Hilbert space. We prove some inequalities about different quantum discords, and also derive an equality which relates one of these quantum discords and entanglement of formation (EOF). Finally, a definition of the quantum discord under two-side measurements is given.

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Introduction: Quantum discord under projective measurements.— Quantum correlation is one of the most striking features in quantum many-body systems. Entanglement was widely regarded as nonlocal quantum correlation and it leads to powerful applications like quantum cryptography, dense coding, and quantum computing [1, 2]. However, entanglement is not the only type of correlation useful for quantum technology. A different notion of measure, quantum discord, has also been proposed to characterize quantum correlation based on quantum measurements [3, 4]. Quantum discord captures the nonlocal correlation more general than entanglement, it can exist in some states even if entanglement does vanish. Moreover, it was shown that quantum discord might be responsible for the quantum computational efficiency of some quantum computation tasks [5–7].

Recently, quantum discord has received much more attention. Its evaluation involves optimization procedure, and analytical expressions are known only in a few cases [8, 9]. A witness of quantum discord for $2 \times n$ states was found [10], while we have known that almost all quantum states have nonvanishing quantum discord [11]. From the theoretical point of view, the relations between quantum discord and other concepts have been discussed, such as Maxwell's demon [12, 13], completely positive maps [14], and relative entropy [15]. Also, the characteristics of quantum discord in some physical models and in information processing have been studied [16–18].

The original definition of quantum discord was defined under projective measurements, in this letter, we give some generalizations of it. This is meaningful not only in mathematics but also in physics since through the generalizations we will get a more fundamental understanding about quantum discord. In particular, one of the generalized quantum discords in this letter has an equality with entanglement of formation, although they are conceptually different measures of quantum correlation.

For clarity, we first give some notations and rules which will be used throughout this letter: Let H^A, H^B be the Hilbert spaces of quantum systems A, B , $\dim H^A = n_A$,

$\dim H^B = n_B$. I_A, I_B are the identity operators on H^A, H^B . The reduced density matrices of a state ρ^{AB} on $H^A \otimes H^B$ are $\rho^A = \text{tr}_B \rho^{AB}$, $\rho^B = \text{tr}_A \rho^{AB}$. For any density operators ρ, σ on a Hilbert space H , the entropy of ρ is $S(\rho) = -\text{tr} \rho \log \rho$ ($\log \rho = \log_2 \rho$), the relative entropy is $S(\rho||\sigma) = \text{tr} \rho \log \rho - \text{tr} \rho \log \sigma$. It is known that $S(\rho||\sigma) \geq 0$ and $S(\rho||\sigma) = 0$ only if $\rho = \sigma$. The conditional entropy of ρ^{AB} (with respect to A) is defined as $S(\rho^{AB}) - S(\rho^A)$, and the mutual information of ρ^{AB} is $S(\rho^A) + S(\rho^B) - S(\rho^{AB})$ which is nonnegative and vanishing only when $\rho^{AB} = \rho^A \otimes \rho^B$. A general measurement on ρ^{AB} is denoted by a set of operators $\{\Phi_\alpha\}_\alpha$ on $H^A \otimes H^B$ satisfying $\sum_\alpha \Phi_\alpha \Phi_\alpha^\dagger = I_A \otimes I_B$ where \dagger means Hermitian adjoint, and $\{\Phi_\alpha\}_\alpha$ operate ρ^{AB} as $\widetilde{\rho}^{AB} = \sum_\alpha \Phi_\alpha \rho^{AB} \Phi_\alpha^\dagger$. When $\Phi_\alpha = A_\alpha \otimes I_B$, where A_α are operators on H^A , we say $\{A_\alpha \otimes I_B\}_\alpha$ is a one-side (with respect to subsystem A) general measurement. Moreover, if $A_\alpha = |\alpha\rangle\langle\alpha|$ and $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ is an orthonormal basis of H^A , we call $\{|\alpha\rangle\langle\alpha| \otimes I_B\}_{\alpha=1}^{n_A}$ a one-side projective measurement. For simplicity, we sometimes write $\sum_\alpha A_\alpha \otimes I_B \rho^{AB} A_\alpha^\dagger \otimes I_B = \sum_\alpha A_\alpha \rho^{AB} A_\alpha^\dagger$ by omitting identity operators. In this letter, we use $\widetilde{\rho}^{AB}$ to denote the state whose initial state are ρ^{AB} and experienced a measurement, and $\widetilde{\rho}^A = \text{tr}_B \widetilde{\rho}^{AB}$, $\widetilde{\rho}^B = \text{tr}_A \widetilde{\rho}^{AB}$. When a third system C is concerned, the notations are similarly extended to C.

Now recall that the quantum discord of ρ^{AB} under projective measurements on A can be expressed as

$$D_A^P(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_P \sum_{\alpha=1}^{n_A} p_\alpha S(\widetilde{\rho}_\alpha^B). \quad (1)$$

In Eq. (1), inf is taken over all projective measurements on A. That is, for all orthonormal bases $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ of H^A , $\widetilde{\rho}_\alpha^B = \frac{1}{p_\alpha} \text{tr}_A(|\alpha\rangle\langle\alpha| \otimes I_B \rho^{AB} |\alpha\rangle\langle\alpha| \otimes I_B)$ are density operators on H^B , and $p_\alpha = \text{tr}_B \text{tr}_A(|\alpha\rangle\langle\alpha| \otimes I_B \rho^{AB} |\alpha\rangle\langle\alpha| \otimes I_B)$ are probabilities. The intuitive meaning of Eq. (1) is that $D_A^P(\rho^{AB})$ is the minimal loss of conditional entropy or mutual information (since $\rho^B = \widetilde{\rho}^B$) under all projective measurements on subsystem A.

$D_A^P(\rho^{AB}) = 0$ means there is no loss of conditional entropy or mutual information for at least one projective measurement on A. Such states are called classical states because of this classical feature. It can be proved that

$$D_A^P(\rho^{AB}) = 0 \iff \rho^{AB} = \sum_{\alpha=1}^{n_A} p_\alpha |\alpha\rangle\langle\alpha| \otimes \rho_\alpha^B, \quad (2)$$

where, $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ is an arbitrary orthonormal set of H^A , and p_α are probabilities.

Although the set of all states ρ^{AB} satisfying $D_A^P(\rho^{AB}) = 0$ is not a convex set, a technical definition of geometric measure of quantum discord of ρ^{AB} under projective measurements on A can be defined as

$$D_A^G(\rho^{AB}) = \inf_{\sigma^{AB}} d(\rho^{AB}, \sigma^{AB}), \quad (3)$$

where d is a distance defined on density operators of $H^A \otimes H^B$, and \inf is taken over all σ^{AB} with $D_A^P(\sigma^{AB}) = 0$. Few analytical expressions and a tight bound for one of such geometric measures have been derived [19, 20].

Quantum discord under one-side general measurements.— To define the quantum discord under one-side general measurements, we need a quantity which is non-negative under all one-side general measurements, and when it comes to the case of one-side projective measurements it can recover the original definition. We define

$$D_A(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{\{A_\alpha\}_\alpha} [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A})]. \quad (4)$$

In Eq.(4) \inf is taken over all general measurements on system A, i.e., $\widetilde{\rho^{AB}} = \sum_\alpha A_\alpha \otimes I_B \rho^{AB} A_\alpha^\dagger \otimes I_B$ with $\sum_\alpha A_\alpha A_\alpha^\dagger = I_A$. We now prove that Eq. (4) is non-negative, and it returns to Eq. (1) under one-side projective measurements on A. From the monotonicity of relative entropy under general measurements [21]

$$S(\sum_\mu \Phi_\mu \rho^{AB} \Phi_\mu^\dagger || \sum_\mu \Phi_\mu \sigma^{AB} \Phi_\mu^\dagger) \leq S(\rho^{AB} || \sigma^{AB}),$$

and the relation between conditional entropy and relative entropy

$$S(\rho^{AB} || \rho^A \otimes \frac{I_B}{n_B}) = S(\rho^A) - S(\rho^{AB}) + \log n_B,$$

combining $\Phi_\mu = A_\mu \otimes I_B$ and $\widetilde{\rho^A \otimes \frac{I_B}{n_B}} = \widetilde{\rho^A} \otimes \frac{I_B}{n_B}$, we can surely get that the right hand side of Eq. (4) is non-negative. When in the case of projective measurements on A, we apply the joint entropy theorem

$$S(\sum_\alpha p_\alpha |\alpha\rangle\langle\alpha| \otimes \rho_\alpha^B) = S(p_\alpha) + \sum_\alpha p_\alpha S(\rho_\alpha^B),$$

where $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ is an orthonormal basis for H^A , $\{p_\alpha\}_{\alpha=1}^{n_A}$ are probabilities, and $S(p_\alpha) = -\sum_\alpha p_\alpha \log p_\alpha$. Note that $S(|\alpha\rangle\langle\alpha|) = 0$, thus Eq. (4) readily returns to Eq. (1).

Similar to Eq. (1), the intuitive meaning of Eq. (4) is that $D_A(\rho^{AB})$ is the minimal loss of conditional entropy or mutual information (since $\rho^B = \widetilde{\rho^B}$) under all general measurements on A.

As a special case of Eq. (4), we consider a subset GP (or $(GP)_A$) of the set of all general measurements on A,

$$GP = \{ \{ \frac{|\beta\rangle\langle\beta|}{\sqrt{p_\beta}} \}_\beta : |\beta\rangle \in H^A, \sum_\beta |\beta\rangle\langle\beta| = I_A, p_\beta = \langle\beta|\beta\rangle \}, \quad (5)$$

and we define

$$D_A^{GP}(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{GP} [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A})], \quad (6)$$

where \inf is taken over all elements of GP. Note that in Eq. (5) $\{|\beta\rangle\}_\beta$ are not necessarily orthogonal and not necessarily normalized. Obviously, GP can be viewed as a generalization of the set of all projective measurements.

The optimization of Eq. (4) is not an easy thing, but we would like to give an upper bound of it. Actually, $S(\rho^A) - S(\rho^{AB}) + \sup [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A})] = S(\rho^A) - S(\rho^{AB}) + S(\rho^B)$, this is just the mutual information. To make clear this assertion, note that $S(\rho^{AB}) - S(\widetilde{\rho^A}) \leq S(\rho^B)$, and there exists a set of unitary matrices U_j on H^A and probabilities p_j such that $\sum_j U_j \otimes I_B \rho^{AB} U_j^\dagger \otimes I_B = \frac{I_A}{n_A} \otimes \rho^B$, this measurement exactly achieves $S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A}) = S(\rho^B)$.

Quantum discord under projective measurements on an extended Hilbert space.— We now generalize Eq. (1) in another way. To do this, we first extend H^A to a countable infinite dimensional Hilbert space \overline{H}_E^A (direct-sum extension) as follows: extend H^A to H^{n_A+1} by adding a vector to H^A such that $\dim H^{n_A+1} = n_A + 1$; extend H^{n_A+1} to H^{n_A+2} by adding a vector to H^{n_A+1} such that $\dim H^{n_A+2} = n_A + 2$; \dots . Then we get a sequence $H^A \subset H^{n_A+1} \subset H^{n_A+2} \subset \dots$. Let $H_E^A = \cup_{n_A \leq N < \infty} H^N$, and $\overline{H}_E^A = \cup_{n_A \leq N \leq \infty} H^N$. (Strictly speaking, \overline{H}_E^A is a Hilbert space, but H_E^A is not, because H_E^A is not “complete”, but \overline{H}_E^A is the “completion” of H_E^A .) We define the set PE (or $(PE)_A$) as

$$PE = \{ \{ |\gamma\rangle\langle\gamma| \}_{\gamma=1}^N : \text{all } N \text{ that } \{ |\gamma\rangle \}_{\gamma=1}^N \text{ is an orthonormal basis for } H^N, H^A \subset H^N \subset H_E^A \}, \quad (7)$$

and define

$$\begin{aligned} D_A^{PE}(\rho^{AB}) &= S(\rho^A) - S(\rho^{AB}) + \inf_{PE} [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A})] \\ &= S(\rho^A) - S(\rho^{AB}) + \inf_{PE} [\sum_{\gamma=1}^N p_\gamma S(\widetilde{\rho^B})]. \end{aligned} \quad (8)$$

In Eq. (8), ρ^{AB} is on $H^N \otimes H^B$, $p_\gamma = \text{tr}_B \langle \gamma | \rho^{AB} | \gamma \rangle = \langle \gamma | \rho^A | \gamma \rangle$, and \inf is taken over the set PE.

In Eq. (8) it does not matter which number N starts from. Suppose $n_A \leq N_1 < N_2 < \infty$, and $\{ |\gamma\rangle \}_{\gamma=1}^{N_1}$ is an

orthonormal basis for H^{N_1} , then there exists $\{|\gamma\rangle\}_{\gamma=N_1+1}^{N_2}$ such that $\{|\gamma\rangle\}_{\gamma=1}^{N_2}$ is an orthonormal basis for H^{N_2} . $\{|\gamma\rangle\}_{\gamma=N_1+1}^{N_2}$ and H^A are disjoint, so

$$\sum_{\gamma=1}^{N_2} |\gamma\rangle\langle\gamma| \rho^{AB} |\gamma\rangle\langle\gamma| = \sum_{\gamma=1}^{N_1} |\gamma\rangle\langle\gamma| \rho^{AB} |\gamma\rangle\langle\gamma|.$$

That is, the value of $[S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A})]$ under $\{|\gamma\rangle\langle\gamma|\}_{\gamma=1}^{N_1}$ can be achieved by $\{|\gamma\rangle\langle\gamma|\}_{\gamma=1}^{N_2}$. Therefore, N starts from N_1 is equivalent to that N starts from N_2 .

Similar to Eq. (2), we have $D_A^{PE}(\rho^{AB}) = 0 \iff \rho^{AB} = \sum_{\gamma} p_{\gamma} |\gamma\rangle\langle\gamma| \otimes \rho_{\gamma}^B$, where $\{|\gamma\rangle\}_{\gamma}$ is an arbitrary orthonormal set in H_E^A , but it is easy to verify that $\{|\gamma\rangle\}_{\gamma}$ is actually in H^A . Consequently, $D_A^{PE}(\rho^{AB}) = 0 \iff D_A^P(\rho^{AB}) = 0$.

Up to now, We have different quantum discords due to different measurements. Then how about their differences or relations? We prove the following proposition.

proposition 1.—The quantum discords $D_A(\rho^{AB})$, $D_A^{GP}(\rho^{AB})$, $D_A^{PE}(\rho^{AB})$, $D_A^P(\rho^{AB})$ of a bipartite state ρ^{AB} defined in Eqs. (4), (6), (8), (1), hold that

$$\begin{aligned} D_A(\rho^{AB}) &\leq D_A^{GP}(\rho^{AB}), \\ D_A^{PE}(\rho^{AB}) &\leq D_A^{GP}(\rho^{AB}) \leq D_A^P(\rho^{AB}). \end{aligned} \quad (9)$$

Proof.—we only need to prove $D_A^{GP}(\rho^{AB}) \leq D_A^{PE}(\rho^{AB})$, the others obviously hold. First note that for any orthonormal basis of H^N , $\{|\gamma\rangle\}_{\gamma=1}^N \subset H^N$, when restrict it to H^A , we obtain $\{|\bar{\gamma}\rangle\}_{\gamma=1}^N \subset H^A$, and $\sum_{\gamma=1}^N |\bar{\gamma}\rangle\langle\bar{\gamma}| = I_A$, where $|\bar{\gamma}\rangle$ is the projection of $|\gamma\rangle$ to H^A . This means that for any element $\{|\gamma\rangle\langle\gamma|\}_{\gamma=1}^N$ of PE , we can get an (unique) element $\{\frac{1}{\sqrt{p_{\bar{\gamma}}}} |\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^N$ of GP through restricting it to H^A , where $p_{\bar{\gamma}} = \langle\bar{\gamma}|\bar{\gamma}\rangle$. Conversely, for any element of GP , we can always extend it to an element of PE (always not unique!). Then from the concavity of conditional entropy, we have

$$\begin{aligned} &S\left(\sum_{\gamma=1}^N \frac{1}{p_{\bar{\gamma}}} |\bar{\gamma}\rangle\langle\bar{\gamma}| \rho^{AB} |\bar{\gamma}\rangle\langle\bar{\gamma}|\right) - S\left(\sum_{\gamma=1}^N \frac{1}{p_{\bar{\gamma}}} |\bar{\gamma}\rangle\langle\bar{\gamma}| \rho^A |\bar{\gamma}\rangle\langle\bar{\gamma}|\right) \\ &\geq \sum_{\gamma=1}^N \langle\bar{\gamma}| \rho^A |\bar{\gamma}\rangle S\left(\frac{|\bar{\gamma}\rangle\langle\bar{\gamma}| \rho^{AB} |\bar{\gamma}\rangle\langle\bar{\gamma}|}{p_{\bar{\gamma}} \langle\bar{\gamma}| \rho^A |\bar{\gamma}\rangle}\right) - \sum_{\gamma=1}^N \langle\bar{\gamma}| \rho^A |\bar{\gamma}\rangle S\left(\frac{|\bar{\gamma}\rangle\langle\bar{\gamma}|}{p_{\bar{\gamma}}}\right) \\ &= \sum_{\gamma=1}^N \langle\gamma| \rho^A |\gamma\rangle S\left(\frac{|\gamma\rangle\langle\gamma| \rho^{AB} |\gamma\rangle\langle\gamma|}{\langle\gamma| \rho^A |\gamma\rangle}\right). \end{aligned}$$

This tells us $D_A^{GP}(\rho^{AB}) \geq D_A^{PE}(\rho^{AB})$, where we have used $\langle\bar{\gamma}| \rho^A |\bar{\gamma}\rangle = \langle\gamma| \rho^A |\gamma\rangle = \text{tr}_B \langle\gamma| \rho^{AB} |\gamma\rangle$, and $S(\frac{|\bar{\gamma}\rangle\langle\bar{\gamma}|}{p_{\bar{\gamma}}}) = 0$. Then we complete the proof.

As a result of proposition 1 and $D_A^{PE}(\rho^{AB}) = 0 \iff D_A^P(\rho^{AB}) = 0$, we have the following proposition.

proposition 2.—The quantum discords $D_A^P(\rho^{AB})$, $D_A^{GP}(\rho^{AB})$, $D_A^{PE}(\rho^{AB})$ of a bipartite state ρ^{AB} defined in Eqs. (1), (6), (8), hold that

$$D_A^{PE}(\rho^{AB}) = 0 \iff D_A^{GP}(\rho^{AB}) = 0 \iff D_A^P(\rho^{AB}) = 0. \quad (10)$$

Relation between quantum discord and entanglement of formation (EOF).— We now prove a theorem which states that there exists a relation between EOF and the quantum discord D_A^{PE} . The theorem concerns a tripartite pure state ρ^{ABC} , and we investigate the relation of the EOF of ρ^{BC} , $E(\rho^{BC})$, and the quantum discord of ρ^{AB} , $D_A^{PE}(\rho^{AB})$. Through this “purification procedure” we can relate these two quantities. This approach was also used in Ref. [22].

Theorem.— Given a tripartite pure state $\rho^{ABC} = |\psi\rangle\langle\psi|$ of a joint system ABC, we have

$$D_A^{PE}(\rho^{AB}) = E(\rho^{BC}) + S(\rho^A) - S(\rho^{AB}). \quad (11)$$

where $E(\rho^{AC})$ is the EOF of ρ^{AC} , and $D_A^{PE}(\rho^{AB})$ is defined in Eq. (8).

Proof.— The EOF of ρ^{AC} is defined as

$$E(\rho^{BC}) = \inf_{\{\sqrt{p_i} |\psi_i^{BC}\rangle\}_i} \sum p_i S(\rho_i^B),$$

where \inf is taken over all pure decompositions $\{\sqrt{p_i} |\psi_i^{BC}\rangle\}_i$ of ρ^{BC} , and $\rho_i^B = \text{tr}_C(|\psi_i^{BC}\rangle\langle\psi_i^{BC}|)$. Recall that if $\{\sqrt{p_i} |\psi_i^{BC}\rangle\}_{i=1}^m$ is a pure decomposition of ρ^{BC} , m is a positive integer, then all pure decompositions of ρ^{BC} are $\cup_{m \leq M < \infty} U_{MM} \{\sqrt{p_i} |\psi_i^{BC}\rangle\}_{i=1}^M$, where $p_i = 0$ for all $i > m$, and U_{MM} is any $M \times M$ unitary matrix. For any M_1, M_2 , with $m < M_1 < M_2 < \infty$, a trivial but useful fact is that any pure decomposition $U_{M_1 M_1} \{\sqrt{p_i} |\psi_i^{BC}\rangle\}_{i=1}^{M_1}$ can be viewed as $U_{M_2 M_2} \{\sqrt{p_i} |\psi_i^{BC}\rangle\}_{i=1}^{M_2}$ through $U_{M_2 M_2} = U_{M_1 M_1} \otimes I_{M_2 - M_1}$, where $I_{M_2 - M_1}$ is the $(M_2 - M_1) \times (M_2 - M_1)$ identity matrix. So, $\cup_{m \leq M < \infty} U_{MM} \{\sqrt{p_i} |\psi_i^{BC}\rangle\}_{i=1}^M = \cup_{m < m' \leq M < \infty} U_{MM} \{\sqrt{p_i} |\psi_i^{BC}\rangle\}_{i=1}^M$ for any integer m' .

The key ingredient of this proof is how we can achieve all pure decompositions of ρ^{BC} through applying measurements on system A. Now using Schmidt decomposition, we write $|\psi\rangle$ as $|\psi\rangle = \sum_{i=1}^n \sqrt{p_i} |\psi_i^A\rangle |\psi_i^{BC}\rangle$, where $p_i \geq 0$, $\sum_{i=1}^n p_i = 1$, $\{|\psi_i^A\rangle\}_{i=1}^n$ is an orthonormal set of H^A , $\{|\psi_i^{BC}\rangle\}_{i=1}^n$ is an orthonormal set of $H^B \otimes H^C$, $n \leq n_A$. Obviously, $\{\sqrt{p_i} |\psi_i^{BC}\rangle\}_{i=1}^n$ is a pure decomposition of ρ^{BC} , hence all pure decompositions of ρ^{BC} are $\cup_{M: n \leq M < \infty} U_{MM} \{\sqrt{p_i} |\psi_i^{BC}\rangle\}_{i=1}^M$.

We now apply a PE measurement $\{|\gamma\rangle\langle\gamma|\}_{\gamma=1}^N$ on system A, $N \geq n_A \geq n$, then

$$\begin{aligned} \widetilde{\rho^{ABC}} &= \sum_{\gamma=1}^N \sum_{i,j=1}^n \sqrt{p_i p_j} |\gamma\rangle\langle\gamma| \psi_i^A \langle\psi_j^A| \gamma\rangle\langle\gamma| \otimes |\psi_i^{BC}\rangle\langle\psi_j^{BC}|, \\ \widetilde{\rho^{BC}} &= \sum_{\gamma=1}^N \left(\sum_{i=1}^n \langle\gamma| \psi_i^A \rangle \sqrt{p_i} |\psi_i^{BC}\rangle \right) \left(\sum_{j=1}^n \langle\psi_j^A| \gamma\rangle \sqrt{p_j} \langle\psi_j^{BC}| \right). \end{aligned}$$

This just realize a pure decomposition of ρ^{BC} , and all $\{|\gamma\rangle\langle\gamma|\}$ of PE will realize all pure decompositions of ρ^{BC} . Combining with Eq. (8) and tracing over the system C, with some direct calculations we will obtain Eq. (11). These complete the proof of the theorem.

In the proof above, we should note that: (i). for $|\psi\rangle = \sum_{i=1}^n \sqrt{p_i} |\psi_i^A\rangle |\psi_i^{BC}\rangle$, the PE measurements can be equivalently replaced by the set

$$\{ \{ |\gamma\rangle\langle\gamma| \}_{\gamma=1}^N : \text{all } N \text{ that } \{ |\gamma\rangle\}_{\gamma=1}^N \text{ is an orthonormal basis for } H^N, H^n \subset H^N \subset H_E^A \},$$

where H^n is the Hilbert space spanned by $\{ |\psi_i^A\rangle \}_{i=1}^n$; (ii). if a PE measurement $\{ |\gamma\rangle\langle\gamma| \}_{\gamma=1}^N$ achieves $D_A^{PE}(\rho^{AB})$, then the pure decomposition $\sum_{\gamma=1}^N \{ \sum_{i=1}^n \langle\gamma|\psi_i^A\rangle \sqrt{p_i} |\psi_i^{BC}\rangle \}_{\gamma=1}^N$ also achieves $E(\rho^{BC})$; conversely, if a pure decomposition $\sum_{\gamma=1}^N \{ \sum_{i=1}^n U_{\gamma i} \sqrt{p_i} |\psi_i^{BC}\rangle \}_{\gamma=1}^N$ achieves $E(\rho^{BC})$, then we can get a PE measurement $\{ |\gamma\rangle\langle\gamma| \}_{\gamma=1}^N$ (always not unique!) which achieves $D_A^{PE}(\rho^{AB})$ and $U_{\gamma i} = \langle\gamma|\psi_i^A\rangle$ for $1 \leq i \leq n$ and $1 \leq \gamma \leq N$.

Eq. (11) established a remarkable connection between EOF and quantum discord (under PE measurements) via the purification procedure. We know that for some special cases the analytical expressions of EOF have been obtained [23], particularly the 2-qubit systems [24]. So according to Eq. (11), we can obtain the corresponding quantum discord of some states. As a demonstration, we consider ρ^{BC} of an arbitrary state of two qubits and $n_A \geq 4$. Suppose the eigen-decomposition of ρ^{BC} is $\rho^{BC} = \sum_{i=1}^4 p_i |\psi_i^{BC}\rangle \langle\psi_i^{BC}|$, we purify it as $|\psi\rangle = \sum_{i=1}^4 \sqrt{p_i} |\psi_i^A\rangle |\psi_i^{BC}\rangle$, where $\{ |\psi_i^A\rangle \}_{i=1}^4$ is an orthonormal set in H^A . But if we use the Schmidt decomposition to the bipartite system in which we regard AB as one system, $|\psi\rangle$ shall be written as $|\psi\rangle = \sum_{i=1}^2 \sqrt{q_i} |\psi_i^{AB}\rangle |\psi_i^C\rangle$, where $\{ |\psi_i^{AB}\rangle \}_{i=1}^2$ is an orthonormal set in $H^A \otimes H^B$, $\{ |\psi_i^C\rangle \}_{i=1}^2$ is an orthonormal set in H^C , $\{ q_i \}_{i=1}^2$ are probabilities. So $\rho^{AB} = \sum_{i=1}^2 q_i |\psi_i^{AB}\rangle \langle\psi_i^{AB}|$, and $\text{rank} \rho^{AB} \leq 2$. Thus, the $D_A^{PE}(\rho^{AB})$ of any $n \times 2$ bipartite state ρ^{AB} with rank no more than 2 can be obtained by this approach. Moreover, since $E(\rho^{BC})$ of any 2-qubit state ρ^{BC} can be achieved by a 4-vector pure decomposition $\{ \sum_{i=1}^4 U_{\gamma i} \sqrt{p_i} |\psi_i^{BC}\rangle \}_{\gamma=1}^4$ of ρ^{BC} , then $D_A^{PE}(\rho^{AB})$ can be achieved by a PE measurement of the form $\{ |\gamma\rangle\langle\gamma| \}_{\gamma=1}^4$.

Conclusions and discussions.— In summary, we generalized the original definition of quantum discord in two ways, and proved some inequalities of different quantum discords and an equality between one of these quantum discords and entanglement of formation. We point out that the definition $D_A^{PE}(\rho^{AB})$ (so does $D_A^P(\rho^{AB})$) can be generalized to the case of two-side PE measurements $(PE)_A \otimes (PE)_B$ as

$$D_{AB}^{PE}(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}) + \inf_{(PE)_A \otimes (PE)_B} [S(\rho^{\widetilde{AB}}) - S(\rho^{\widetilde{A}}) - S(\rho^{\widetilde{B}})] \quad (12)$$

Recall that $S(\rho^{\widetilde{AB}}) - S(\rho^{\widetilde{A}}) - S(\rho^{\widetilde{B}}) = -S(\rho^{AB} || \rho^A \otimes \rho^B)$, and note that $\rho^A \otimes \rho^B = \rho^{\widetilde{A}} \otimes \rho^{\widetilde{B}}$ under $(PE)_A \otimes (PE)_B$,

then use the similar techniques in the proof about Eq. (4), we will find that the right hand side of Eq. (12) is non-negative.

There remains an interesting question to consider: the physical interpretation of PE measurements. The states on H^A under a general measurement are still on H^A , but under a PE measurement which will be on the space H_E^A . We may ask: the PE measurements are only mathematical conveniences or being of physical reality—i.e., are quantum systems intrinsically infinite dimensional?

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Generalizations of quantum discord

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Abstract. The original definition of quantum discord of bipartite states was defined over projective measurements, in this paper we discuss some generalizations of it. These generalizations are defined over general measurements, rank-one general measurements or Neumark extension measurements. We investigate the nonnegativity, zero-discord sets of all these quantum discords and some properties about them.

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1. Introduction: quantum discord over projective measurements

Quantum correlation is one of the most striking features in quantum many-body systems. Entanglement was widely regarded as nonlocal quantum correlation and it leads to powerful applications [1, 2]. However, entanglement is not the only type of correlation useful for quantum technology. A different notion of measure, quantum discord, has also been proposed to characterize quantum correlation based on quantum measurements [3, 4]. Quantum discord captures the nonlocal correlation more general than entanglement, it can exist in some states even if entanglement does vanish. Moreover, it was shown that quantum discord might be responsible for the quantum computational efficiency of some quantum computation tasks [5, 6, 7].

Recently, quantum discord has attracted increasing attention. Its evaluation involves optimization procedure, and analytical expressions are known only for very few cases [8, 9, 10]. A witness of quantum discord for $2 \times n$ states was found [11], while we have known that almost all quantum states have nonvanishing quantum discord [12]. Theoretically, the relations between quantum discord and other concepts have been discussed, such as Maxwell's demon [13, 14], completely positive maps [15], and relative entropy [16]. Also, the characteristics of quantum discord in some physical models and in information processing have been studied [17, 18, 19, 20]. An interesting geometric measure of quantum discord was introduced [21] and discussed [22]. Very recently, operational interpretations of quantum discord were proposed [23, 24],

The original definition of quantum discord was given over projective measurements. In this paper, we discuss some generalizations of it. These generalizations will be defined over more extensive measurements than projective measurements. For clarity, we first give some notations which will be used throughout this paper. Let H^A, H^B be the Hilbert spaces of quantum systems A, B , respectively, with $\dim H^A = n_A, \dim H^B = n_B$. I_A, I_B are the identity operators on H^A and H^B . The reduced density matrices of a state ρ^{AB} on $H^A \otimes H^B$ are $\rho^A = \text{tr}_B \rho^{AB}$, $\rho^B = \text{tr}_A \rho^{AB}$. For any density operators ρ, σ on a Hilbert space H , the entropy of ρ is $S(\rho) = -\text{tr}(\rho \log \rho)$ ($\log \rho = \log_2 \rho$), the relative entropy is $S(\rho||\sigma) = \text{tr}(\rho \log \rho) - \text{tr}(\rho \log \sigma)$. It is known that $S(\rho||\sigma) \geq 0$ and $S(\rho||\sigma) = 0$ only if $\rho = \sigma$ ([1], 11.3.1). The conditional entropy of ρ^{AB} on $H^A \otimes H^B$ (with respect to A) is defined as $S(\rho^{AB}) - S(\rho^A)$. The mutual information of ρ is $S(\rho^A) + S(\rho^B) - S(\rho^{AB})$, which is nonnegative and vanishing only when $\rho^{AB} = \rho^A \otimes \rho^B$ ([1], 11.3.4). A general measurement on ρ^{AB} is denoted by a set of operators $\Phi = \{\Phi_\alpha\}_\alpha$ satisfying $\sum_\alpha \Phi_\alpha^\dagger \Phi_\alpha = I_A \otimes I_B$, where \dagger means Hermitian adjoint, and $\{\Phi_\alpha\}_\alpha$ operates ρ^{AB} as $\widetilde{\rho}^{AB} = \sum_\alpha \Phi_\alpha \rho^{AB} \Phi_\alpha^\dagger$. When $\Phi_\alpha = A_\alpha \otimes I_B$, where A_α are operators on H^A , we say $\{A_\alpha \otimes I_B\}_\alpha$ is a one-sided (with respect to A) general measurement. Moreover, if $A_\alpha = \Pi_\alpha = |\alpha\rangle\langle\alpha|$ and $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ is an orthonormal basis of H^A , we call $\{\Pi_\alpha \otimes I_B\}_\alpha$ a one-sided projective measurement. We sometimes simply write $A_\alpha \otimes I_B$ as A_α by omitting identity operators. We use $\widetilde{\rho}^{AB}$ to denote the state whose initial state is ρ^{AB} and experienced a measurement, and $\widetilde{\rho}^A = \text{tr}_B \widetilde{\rho}^{AB}$, $\widetilde{\rho}^B = \text{tr}_A \widetilde{\rho}^{AB}$. When the third system C is concerned, the notations will be similarly extended to it.

Now recall that the original definition of quantum discord of ρ^{AB} was defined over projective measurements (on A) as [3]

$$D_A^P(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{\{\Pi_\alpha \otimes I_B\}_\alpha} [\sum_\alpha p_\alpha S(\widetilde{\rho}_\alpha^B/p_\alpha)]. \quad (1)$$

In Eq.(1), \inf takes over all projective measurements on A, $\widetilde{\rho}_\alpha^B = \text{tr}_A(\Pi_\alpha \rho^{AB} \Pi_\alpha)$, $p_\alpha = \text{tr}_B \rho_\alpha^B$.

Using the joint entropy theorem ([1], 11.3.2), Eq.(1) can also be written as [3]

$$D_A^P(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{\{\Pi_\alpha \otimes I_B\}_\alpha} [S(\widetilde{\rho}^{AB}) - S(\widetilde{\rho}^A)]. \quad (2)$$

A state ρ^{AB} satisfying $D_A(\rho^{AB}) = 0$ is called classical state, it can be proved [3]

$$D_A^P(\rho^{AB}) = 0 \iff \rho^{AB} = \sum_{\alpha=1}^{n_A} p_\alpha |\alpha\rangle\langle\alpha| \otimes \rho_\alpha^B, \quad (3)$$

where, $\{|\alpha\rangle\}_{\alpha=1}^{n_A}$ is an arbitrary orthonormal set of H^A , $p_\alpha \geq 0$, $\sum_\alpha p_\alpha = 1$. ρ_α^B are density operators on H^B .

Eq.(1) or Eq.(2) have intuitive physical meanings, namely, $D_A^P(\rho^{AB})$ is the minimal loss of conditional entropy or mutual information over all projective measurements. To generalize the definition of quantum discord to other measurements, a direct idea is, we define the quantum discord as Eq.(1) or Eq.(2) but let \inf take other measurements. Doing this, we must guarantee the nonnegativity of the definitions like Eq.(1) or Eq.(2) since the positive quantum discord was regarded as a measure of quantum correlation.

The remainder of this paper is arranged as follows. In Sec.II, we consider the generalization of Eq.(2) to general measurements. In Sec.III, we consider the generalization of Eq.(1) to general measurements and the quantum discord defined over Neumark extension measurements. In Sec.IV, we discuss some relations and properties about these quantum discords. Finally, Sec.V is devoted to a brief summary.

2. Generalization of Eq.(2) to general measurements

To generalize Eq.(2) to general measurements, we first prove the theorem below.

Theorem. For any state ρ^{AB} and any general measurement $\{A_\alpha \otimes I_B\}_\alpha$ performing on A, it holds that

$$S(\rho^A) - S(\rho^{AB}) + [S(\widetilde{\rho}^{AB}) - S(\widetilde{\rho}^A)] \geq 0. \quad (4)$$

Proof. Suppose

$$\rho^{AB} = \sum_i c_i \rho_i^A \otimes \rho_i^B, \quad (5)$$

where ρ_i^A , ρ_i^B are Hermitian operators on H^A and H^B , c_i are real numbers (this is a very useful representation for bipartite states, see, e.g., [25]). Performing a general measurement $\{A_\alpha \otimes I_B\}_\alpha$ on ρ^{AB} , we have

$$\rho^A = \text{tr}_B \rho^{AB} = \sum_i c_i \rho_i^A (\text{tr}_B \rho_i^B), \quad (6a)$$

$$\rho^B = \text{tr}_A \rho^{AB} = \sum_i c_i (\text{tr}_A \rho_i^A) \rho_i^B, \quad (6b)$$

$$\widetilde{\rho^{AB}} = \sum_\alpha A_\alpha \rho^{AB} A_\alpha^\dagger = \sum_{\alpha i} c_i A_\alpha \rho_i^A A_\alpha^\dagger \otimes \rho_i^B, \quad (6c)$$

$$\widetilde{\rho_\alpha^A} = \text{tr}_B (A_\alpha \rho^{AB} A_\alpha^\dagger) = A_\alpha \rho^A A_\alpha^\dagger, \quad (6d)$$

$$\widetilde{\rho^A} = \sum_\alpha \widetilde{\rho_\alpha^A} = \sum_\alpha A_\alpha \rho^A A_\alpha^\dagger, \quad (6e)$$

$$\widetilde{\rho_\alpha^B} = \text{tr}_A (A_\alpha \rho^{AB} A_\alpha^\dagger) = \sum_i c_i [\text{tr}_A (A_\alpha \rho_i^A A_\alpha^\dagger)] \rho_i^B, \quad (6f)$$

$$\widetilde{\rho^B} = \sum_\alpha \widetilde{\rho_\alpha^B} = \rho^B, \quad (6g)$$

$$\text{tr}_A \widetilde{\rho_\alpha^A} = \text{tr}_A (A_\alpha \rho^A A_\alpha^\dagger) = \text{tr}_B \widetilde{\rho_\alpha^B}, \quad (6h)$$

$$\rho^A \otimes \frac{I_B}{n_B} = \sum_\alpha A_\alpha \rho^A A_\alpha^\dagger \otimes \frac{I_B}{n_B} = \sum_\alpha \widetilde{\rho_\alpha^A} \otimes \frac{I_B}{n_B}. \quad (6i)$$

In Eq.(6g), we have used $\sum_\alpha \text{tr}_A (A_\alpha^\dagger A_\alpha) = I_A$. Notice that $\widetilde{\rho^{AB}}, \widetilde{\rho^A}, \widetilde{\rho^B}$ are all density operators, but $\widetilde{\rho_\alpha^A}, \widetilde{\rho_\alpha^B}, \rho_i^A, \rho_i^B$ are not necessarily so.

For any density operators ρ^{AB} and σ^{AB} , and any general measurement $\Phi = \{\Phi_\mu\}_\mu$, the monotonicity of relative entropy reads [26]

$$S(\Phi \rho^{AB} || \Phi \sigma^{AB}) \leq S(\rho^{AB} || \sigma^{AB}). \quad (7)$$

Also, conditional entropy and relative entropy have the relation ([1], 11.4.1)

$$S(\rho^{AB} || \rho^A \otimes \frac{I_B}{n_B}) = S(\rho^A) - S(\rho^{AB}) + \log n_B. \quad (8)$$

Now, letting $\sigma^{AB} = \rho^A \otimes \frac{I_B}{n_B}$ and $\Phi = \{A_\alpha \otimes I_B\}_\alpha$ in Eq.(7), combining Eq.(6i) and Eq.(8), we can surely get Eq.(4), then end this proof.

We denote the set of all general measurements on A by G,

$$G = \{\{A_\alpha\}_\alpha : \sum_\alpha A_\alpha^\dagger A_\alpha = I_A\}, \quad (9)$$

and denote the set of all rank-1 general measurements on A by R,

$$R = \{\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n : |\gamma\rangle \in H^A, \sum_\gamma |\gamma\rangle\langle\gamma| = I_A, p_\gamma = \langle\gamma|\gamma\rangle; \text{all } n, n \geq n_A\}. \quad (10)$$

Now from Eq.(2) and Eq.(4), we define

$$D_A^S(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{S \otimes I_B} [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A})], \quad (11)$$

where $S \subset G$. Under this definition, Proposition 1, Proposition 2 and Proposition 3 below are easy to get.

Proposition 1. For any state ρ^{AB} , $D_A^S(\rho^{AB})$ defined as in Eq.(11), then

$$D_A^S(\rho^{AB}) \geq 0. \quad (12)$$

Proposition 2. For any state ρ^{AB} , $D_A^S(\rho^{AB})$ defined as in Eq.(11), then

$$\{I_A\} \in S \Rightarrow D_A^S(\rho^{AB}) = 0. \quad (13)$$

Eq.(13) is true, because $\{I_A\}$ results in the equality in Eq.(4) for any state ρ^{AB} . Since $\{I_A\} \in G$, thus

Proposition 3. For any state ρ^{AB} , $D_A^G(\rho^{AB})$ defined as in Eq.(11), then

$$D_A^G(\rho^{AB}) = 0. \quad (14)$$

So, $D_A^S(\rho^{AB})$ is not trivial only if $\{I_A\} \notin S$, such as S takes the set P (all projective measurements), or the set R (all rank-1 general measurements).

The intuitive meaning of Eq.(11) is that $D_A^S(\rho^{AB})$ is the minimal loss of conditional entropy or mutual information (since $\rho^B = \widetilde{\rho^B}$, see Eq.(6g)) over a set of some general measurements on A .

The optimization of Eq.(11) is not an easy thing in general ($I_A \notin S$), but we would like to give an upper bound of it (although, any general measurement in the set S will yield a corresponding upper bound). Actually, the mutual information of ρ^{AB} is an upper bound of $D_A^S(\rho^{AB})$ for any S and any state ρ^{AB} .

Proposition 4. For any state ρ^{AB} and any set $S \in G$, $D_A^S(\rho^{AB})$ defined as in Eq.(4), it holds that

$$D_A^S(\rho^{AB}) \leq S(\rho^A) + S(\rho^B) - S(\rho^{AB}). \quad (15)$$

To make clear this assertion, note that mutual information is nonnegative and $\rho^B = \widetilde{\rho^B}$ (see Eq.(6g)), then $S(\widetilde{\rho^{AB}}) - S(\rho^A) \leq S(\rho^B)$. Moreover, it is known that there exists a set of unitary matrices U_j on H^A and probabilities p_j such that $\sum_j p_j U_j \otimes I_B \rho^{AB} U_j^\dagger \otimes I_B = \frac{I_A}{n_A} \otimes \rho^B$ ([1], 11.3.4). The measurement $\{\sqrt{p_j} U_j \otimes I_B\}_j$ exactly achieves $S(\widetilde{\rho^{AB}}) - S(\rho^A) = S(\rho^B)$. This implies the equality in Eq.(15) can be achieved for some set S .

3. Generalization of Eq.(1) to general measurements

We now consider the generalization of Eq.(1) to general measurements as

$$\overline{D}_A^S(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{S \otimes I_B} [\sum_{\alpha} p_{\alpha} S(\widetilde{\rho_{\alpha}^B}/p_{\alpha})], \quad (16)$$

where $S \subset G$, $p_{\alpha} = \text{tr}_B \widetilde{\rho_{\alpha}^B}$, $\widetilde{\rho_{\alpha}^B}$ specified in Eq.(6f). We need to prove $\overline{D}_A^S(\rho^{AB}) \geq 0$. To do this, we first point out that

$$[\sum_{\alpha} p_{\alpha} S(\widetilde{\rho_{\alpha}^B}/p_{\alpha})]_{\{A_{\alpha}\}_{\alpha}} \geq [\sum_{\alpha} p_{\alpha} S(\widetilde{\rho_{\alpha}^B}/p_{\alpha})]_{R\{\{A_{\alpha}\}_{\alpha}\}}. \quad (17)$$

In Eq.(17), $[\sum_{\alpha} p_{\alpha} S(\widetilde{\rho_{\alpha}^B}/p_{\alpha})]_{\{A_{\alpha}\}_{\alpha}}$ means $[\sum_{\alpha} p_{\alpha} S(\widetilde{\rho_{\alpha}^B}/p_{\alpha})]$ was defined under the general measurement $\{A_{\alpha}\}_{\alpha}$, and $R\{\{A_{\alpha}\}_{\alpha}\}$ is the rank-1 decomposition of $\{A_{\alpha}\}_{\alpha}$,

$$R\{\{A_{\alpha}\}_{\alpha}\} = \{ \{ \frac{|\alpha_j\rangle\langle\alpha_j|}{\sqrt{\langle\alpha_j|\alpha_j\rangle}} \}_{\alpha,\alpha_j} : A_{\alpha}^{\dagger} A_{\alpha} = \sum_{\alpha_j} |\alpha_j\rangle\langle\alpha_j| \}. \quad (18)$$

In Eq.(18), $A_{\alpha}^{\dagger} A_{\alpha} = \sum_{\alpha_j} |\alpha_j\rangle\langle\alpha_j|$ is the eigendecomposition of the positive operator $A_{\alpha}^{\dagger} A_{\alpha}$. Eq.(17) can be obtained [27, 28] by using the concavity of entropy ([1], 11.3.5)

$$S(\sum_i p_i \rho_i) \geq \sum_i p_i S(\rho_i), \quad (19)$$

where $p_i \geq 0$, $\sum_i p_i = 1$, ρ_i are density operators.

Here we would like to consider another generalization of quantum discord, which defined over Neumark extension measurements.

Neumark extension [29, 30] says any general measurement $\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n$ of \mathcal{R} can be extended to a projective measurement $\{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n$ on H_n^A , here H_n^A is a direct-sum extended Hilbert space of H^A with $\dim H_n^A = n \geq n_A$, and $|\gamma\rangle$ are just the restrictions of $|\bar{\gamma}\rangle$ onto H^A . Evidently, the Neumark extension $\{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n$ for $\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n$ is not necessarily unique, but given any orthonormal basis $\{|\bar{\gamma}\rangle\}_{\gamma=1}^n$ of H_n^A , there is only one $\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n$ that $\{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n$ is its Neumark extension. (Some recent discussions about Neumark extension see [31, 32, 33].) Let

$$N = \{ \{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n : \{|\bar{\gamma}\rangle\}_{\gamma=1}^n \text{ is any orthonormal basis of } H_n^A; \text{ all } n, n \geq n_A \}. \quad (20)$$

We now consider the quantum discord $D_A^N(\rho^{AB})$ over Neumark extension measurements as

$$\overline{D}_A^N(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{N \otimes I_B} [S(\widetilde{\rho^{AB}}) - S(\widetilde{\rho^A})]. \quad (21)$$

Just as the equivalence of Eq.(1) and Eq.(2), Eq.(21) can also be written as the form of Eq.(1).

Suppose a general measurement $\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n$ of \mathcal{R} , $\{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n$ is its Neumark extension, note that $\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n$ is performed on H^A , while $\{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n$ is performed on H_n^A .

Neumark extension measurements are very similar with the projective measurements only performing on a larger space. Since Eq.(4) is hold for a projective measurement, with the similar expressions of Eqs.(6a-6i) for Neumark extension measurements, we steadily get

$$S(\rho^A) - S(\rho^{AB}) + [\sum_{\alpha} p_{\alpha} S(\widetilde{\rho_{\alpha}^B}/p_{\alpha})]_{\{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n} \geq 0. \quad (22)$$

From Eq.(6f), it is easy to find

$$[\sum_{\alpha} p_{\alpha} S(\widetilde{\rho_{\alpha}^B}/p_{\alpha})]_{\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n} = [\sum_{\alpha} p_{\alpha} S(\widetilde{\rho_{\alpha}^B}/p_{\alpha})]_{\{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n}, \quad (23)$$

where $\{|\bar{\gamma}\rangle\langle\bar{\gamma}|\}_{\gamma=1}^n$ is the Neumark extension of rank-1 general measurement $\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n$.

Combining Eq.(17), Eq.(23), Eq.(22), Eq.(16), we obtain proposition 5 and proposition 6 below.

Proposition 5. For any state ρ^{AB} , $\overline{D}_A^S(\rho^{AB})$ defined as in Eq.(16), then

$$\overline{D}_A^S(\rho^{AB}) \geq 0, \quad (24)$$

Proposition 6. For any state ρ^{AB} , $\overline{D}_A^S(\rho^{AB})$ defined as in Eq.(16), then

$$\overline{D}_A^G(\rho^{AB}) = \overline{D}_A^R(\rho^{AB}) = \overline{D}_A^N(\rho^{AB}). \quad (25)$$

Eq.(23) and Eq.(25) will be used frequently in next section.

4. Some properties about different quantum discords

We prove that

Proposition 7. The quantum discords $D_A^P(\rho^{AB})$, $D_A^R(\rho^{AB})$, $\overline{D}_A^R(\rho^{AB})$ of a bipartite state ρ^{AB} defined in Eq.(1), Eq.(11) and Eq.(16), hold that

$$D_A^P(\rho^{AB}) \geq D_A^R(\rho^{AB}) \geq \overline{D}_A^R(\rho^{AB}). \quad (26)$$

Proof. We only need to prove $D_A^R(\rho^{AB}) \geq \overline{D}_A^N(\rho^{AB})$. First note that for any $\{|\overline{\gamma}\rangle\langle\overline{\gamma}|\}_{\gamma=1}^n \in N$, when restrict it onto H^A , we obtain $\{\frac{|\gamma\rangle\langle\gamma|}{\sqrt{p_\gamma}}\}_{\gamma=1}^n \in R$, where $|\gamma\rangle$ is the projection of $|\overline{\gamma}\rangle$ onto H^A . According to the definitions of $D_A^R(\rho^{AB})$ and $\overline{D}_A^N(\rho^{AB})$, we have

$$D_A^R(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{R \otimes I_B} [S(\sum_{\gamma=1}^n \frac{1}{p_\gamma} |\gamma\rangle\langle\gamma| \rho^{AB} |\gamma\rangle\langle\gamma|) - S(\sum_{\gamma=1}^n \frac{1}{p_\gamma} |\gamma\rangle\langle\gamma| \rho^A |\gamma\rangle\langle\gamma|)],$$

$$\overline{D}_A^N(\rho^{AB}) = S(\rho^A) - S(\rho^{AB}) + \inf_{N \otimes I_B} \sum_{\gamma=1}^n \langle\overline{\gamma}|\rho^A|\overline{\gamma}\rangle S(\frac{\langle\overline{\gamma}|\rho^{AB}|\overline{\gamma}\rangle}{\langle\overline{\gamma}|\rho^A|\overline{\gamma}\rangle}).$$

Then from the concavity of conditional entropy ([1], 11.4.1), we have

$$\begin{aligned} & S(\sum_{\gamma=1}^n \frac{1}{p_\gamma} |\gamma\rangle\langle\gamma| \rho^{AB} |\gamma\rangle\langle\gamma|) - S(\sum_{\gamma=1}^n \frac{1}{p_\gamma} |\gamma\rangle\langle\gamma| \rho^A |\gamma\rangle\langle\gamma|) \\ & \geq \sum_{\gamma=1}^n \langle\gamma|\rho^A|\gamma\rangle S(\frac{|\gamma\rangle\langle\gamma| \rho^{AB} |\gamma\rangle\langle\gamma|}{p_\gamma \langle\gamma|\rho^A|\gamma\rangle}) - \sum_{\gamma=1}^n \langle\gamma|\rho^A|\gamma\rangle S(\frac{|\gamma\rangle\langle\gamma|}{p_\gamma}) \\ & = \sum_{\gamma=1}^n \langle\overline{\gamma}|\rho^A|\overline{\gamma}\rangle S(\frac{|\overline{\gamma}\rangle\langle\overline{\gamma}| \rho^{AB} |\overline{\gamma}\rangle\langle\overline{\gamma}|}{\langle\overline{\gamma}|\rho^A|\overline{\gamma}\rangle}). \end{aligned}$$

Where we have used $p_\gamma = \langle\gamma|\gamma\rangle$, $\langle\gamma|\rho^A|\gamma\rangle = \langle\overline{\gamma}|\rho^A|\overline{\gamma}\rangle = \text{tr}_B \langle\gamma|\rho^{AB}|\gamma\rangle = \text{tr}_B \langle\overline{\gamma}|\rho^{AB}|\overline{\gamma}\rangle$, $S(\frac{|\gamma\rangle\langle\gamma|}{p_\gamma}) = 0$, and $S(\frac{|\overline{\gamma}\rangle\langle\overline{\gamma}| \rho^{AB} |\overline{\gamma}\rangle\langle\overline{\gamma}|}{\langle\overline{\gamma}|\rho^A|\overline{\gamma}\rangle}) = S(\frac{\langle\overline{\gamma}|\rho^{AB}|\overline{\gamma}\rangle}{\langle\overline{\gamma}|\rho^A|\overline{\gamma}\rangle})$. This leads to Eq.(26), and end the proof.

To find the states for $D_A^R(\rho^{AB}) = 0$ or $\overline{D}_A^R(\rho^{AB}) = 0$, we make a digression to introduce an elegant result in [28]. Given a tripartite pure state $\rho^{ABC} = |\psi\rangle\langle\psi|$ of a joint system ABC, by Schmidt decomposition ([1], 2.5) we write ρ^{ABC} as

$$\rho^{ABC} = \sum_{i,j=1}^m \sqrt{p_i p_j} |\psi_i^A\rangle\langle\psi_j^A| \otimes |\psi_i^{BC}\rangle\langle\psi_j^{BC}|, \quad (27)$$

where $m = \text{rank} \rho^{BC} \leq n_A$, $p_i > 0$, $\sum_i p_i = 1$, $\{|\psi_i^A\rangle\}$, $\{|\psi_i^{BC}\rangle\}$ are orthonormal sets of H^A and $H^B \otimes H^C$, respectively. Performing a Neumark extension measurement $\{|\overline{\gamma}\rangle\langle\overline{\gamma}|\}_{\gamma=1}^n$ on system A, then

$$\widetilde{\rho}_\gamma^B = \text{tr}_C [(\sum_{i=1}^n \langle\overline{\gamma}|\psi_i^A\rangle \sqrt{p_i} |\psi_i^{BC}\rangle)(\sum_{j=1}^n \langle\psi_j^A|\overline{\gamma}\rangle \sqrt{p_j} \langle\psi_j^{BC}|)]. \quad (28)$$

Notice that $\{\sum_{i=1}^n \langle\overline{\gamma}|\psi_i^A\rangle \sqrt{p_i} |\psi_i^{BC}\rangle\}_\gamma$ is just a pure state decomposition of ρ^{BC} ([1], 2.4.2), and all Neumark extension measurements realize all pure state decompositions of ρ^{BC} , this leads to the result in [28]

$$E(\rho^{BC}) = \inf_{R \otimes I_B} \sum_\alpha p_\alpha S(\widetilde{\rho}_\alpha^B / p_\alpha), \quad (29)$$

where $E(\rho^{BC})$ is the entanglement of formation (EOF) [34] of ρ^{BC} .

Here, we make an explanation. From Eq.(27), $\rho^{BC} = \sum_{i=1}^m p_i |\psi_i^{BC}\rangle\langle\psi_i^{BC}|$ is the eigendecomposition of ρ^{BC} , so all pure state decompositions of ρ^{BC} are $F = \cup_{N \geq m} F_N$, where ([1], 2.4.2)

$$F_N = \left\{ \left\{ \sum_{i=1}^N U_{\lambda i} p_i |\psi_i^{BC}\rangle \right\}_{\lambda=1}^N : p_i = 0 \text{ for } i \geq m; U = (U_{\lambda i}) \text{ is any } N \times N \text{ unitary matrix} \right\}.$$

Note that if $m \leq N \leq N_1$, then $F_N \subset F_{N_1}$. This is true since any $N \times N$ unitary matrix multiplied by a $(N_1 - N) \times (N_1 - N)$ identity matrix becomes an $N_1 \times N_1$ unitary matrix. Since $n \geq n_A \geq m$, that is why all Neumark extension measurements realize all pure state decompositions of ρ^{BC} in Eq.(28).

Now if we add $S(\rho^A) - S(\rho^{AB})$ to both sides of Eq.(29), we obtain

$$\overline{D}_A^R(\rho^{AB}) = E(\rho^{BC}) + S(\rho^A) - S(\rho^{AB}). \quad (30)$$

Notice that $\overline{D}_A^R(\rho^{AB})$ and $E(\rho^{BC})$ can be achieved by the same Neumark extension measurement. It is known $E(\rho^{BC})$ can be achieved by a finite l ($l \geq m$) pure decomposition [35], then, correspondingly, $\overline{D}_A^R(\rho^{AB})$ can be achieved by a Neumark extension measurement $\{|\overline{\gamma}\rangle\langle\overline{\gamma}|\}_{\gamma=1}^n$ with finite n . Moreover, if $E(\rho^{BC})$ can be achieved by a finite l ($m \leq l \leq n_A$) pure decomposition, then $\overline{D}_A^R(\rho^{AB})$ can be achieved by a projective measurement, i.e., $\overline{D}_A^R(\rho^{AB}) = D_A^P(\rho^{AB})$. From these facts we obtain proposition 8 and proposition 9 below.

Proposition 8. Quantum discord $D_A^P(\rho^{AB})$ defined in Eq.(1) of any state ρ^{AB} of $n_A \times 2$ systems with rank no more than 2 can be analytically obtained according to Eq. (30).

Proof. Suppose $\rho^{AB} = \sum_{i=1}^2 q_i |\psi_i^{AB}\rangle\langle\psi_i^{AB}|$, $q_i \geq 0$, $\sum_{i=1}^2 q_i = 1$, $\{|\psi_i^{AB}\rangle\}_{i=1}^2$ is an orthonormal set in $H^A \otimes H^B$. We purify ρ^{AB} as $|\psi\rangle = \sum_{i=1}^2 \sqrt{q_i} |\psi_i^{AB}\rangle |\psi_i^C\rangle$, where $\{|\psi_i^C\rangle\}_{i=1}^2$ is an orthonormal set in H^C . At the same time we can also write $|\psi\rangle$ as $|\psi\rangle = \sum_{i=1}^m \sqrt{p_i} |\psi_i^A\rangle |\psi_i^{BC}\rangle$ by schmidt decomposition if we regard BC as one system, where $m = \text{rank} \rho^{BC} \leq \min\{n_A, 4\}$, $p_i \geq 0$, $\sum_{i=1}^4 p_i = 1$, $\{|\psi_i^A\rangle\}_{i=1}^4$ and $\{|\psi_i^{BC}\rangle\}_{i=1}^4$ are orthonormal sets in H^A and $H^B \otimes H^C$ respectively. Because $E(\rho^{BC})$ allows analytical expression for any two qubits state ρ^{BC} [36], and $E(\rho^{BC})$ can be achieved by an m-vector pure decomposition of ρ^{BC} , so $D_A^R(\rho^{AB})$ can be achieved by an m-dimensional projective measurement (on the space spanned by $\{|\psi_i^A\rangle\}_{i=1}^m$) and it can be extended to a projective measurement since $m \leq n_A$. Hence $\overline{D}_A^R(\rho^{AB}) = D_A^P(\rho^{AB})$ and we complete this proof.

Proposition 9. The quantum discords $D_A^P(\rho^{AB})$, $D_A^R(\rho^{AB})$, $\overline{D}_A^R(\rho^{AB})$ of a bipartite state ρ^{AB} defined in Eq.(1), Eq.(11) and Eq.(16), hold that

$$D_A^P(\rho^{AB}) = 0 \iff D_A^R(\rho^{AB}) = 0 \iff \overline{D}_A^R(\rho^{AB}). \quad (31)$$

Proof. Suppose $\overline{D}_A^R(\rho^{AB})$ can be achieved by an element $\{|\overline{\gamma}\rangle\langle\overline{\gamma}|\}_{\gamma=1}^n$ of finite n in the set N . Therefore, similar to Eq. (2), we have

$$\overline{D}_A^N(\rho^{AB}) = 0 \iff \rho^{AB} = \sum_{\gamma=1}^n p_\gamma |\overline{\gamma}\rangle\langle\overline{\gamma}| \otimes \rho_\gamma^B, \quad (32)$$

where $\{|\bar{\gamma}\rangle\}_{\gamma=1}^n$ is an arbitrary orthonormal set in H_n^A , $p_\gamma \geq 0$, $\sum_{\gamma=1}^n p_\gamma = 1$, ρ_γ^B are density operators on H^B . But $\{|\bar{\gamma}\rangle\}_{\gamma=1}^n$ is actually in H^A since ρ^{AB} is on $H^A \otimes H^B$. As a result,

$$\overline{\overline{D}}_A^N(\rho^{AB}) = 0 \iff D_A^P(\rho^{AB}) = 0. \quad (33)$$

Combining Eq.(26), we obtain Eq.(31).

5. Summary

We investigated some generalizations of quantum discord which were defined over general measurements, rank-1 general measurements or Neumark extension measurements. The nonnegativity and zero-discord states were emphasized and some relations about different quantum discords were discussed.

In quantum information and quantum computation, we aim for an exquisite level of control over the measurements, so it is natural to consider the more comprehensive general measurements (such as the optimal way to distinguish a set of quantum states) rather than projective measurements. We expect that these discussions about the generalizations of quantum discord may provide more extensive understandings for characterizations of the nonlocal correlation.

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